The Stable Marriage Problem

Fernando Martinez-Lopez
Graduate School of Arts and Sciences
Fordham University
New York, United States
fmartinezlopez@fordham.edu

***Abstract*— The Stable Marriage or Stable Matching Problem is a game theory problem applied in multiple areas. Solutions to this problem help to match two sets of members who have an interest in each other. The implementation of solutions to this problem has led to enormous advances in our society's development.**

Keywords—match, stable matching, unstable matching, optimal matching, disjoint sets, joint preferences, ranked list.

I. Introduction

Since society's resources are finite, yet human demands are typically boundless; and since any of these resources might have numerous different applications, resource allocation has traditionally been a highly concentrated field of research for multiple disciplines, especially economics, mathematics, and computer science.

There are various considerations when allocating members of two disjoint sets to one another. In economics, the classic assignment issue is the one in which each assignment has a cost attached to it and the goal is to maximize or decrease the overall cost of all the terms, which is the most obvious example. However, this sort of assignment criterion isn't always the most appropriate. There are assignment/matching cases where agents care more about other factors such as with whom they are dealing than economical or resource-wise. This is the case that gave birth to the Stable Marriage Problem.

In 1962, D. Gale and L. S. Shapley approached to solve this problem by developing an algorithm specifically designed for the following problem[1]:

"A college is considering a set of *n* applicants of which it can admit a quota of only *q*. Having evaluated their qualifications, the admissions office must decide which ones to admit. The procedure of offering admission only to the *q* best-qualified applicants will not generally be satisfactory, for it cannot be assumed that all who are offered admission will accept. Accordingly, in order for a college to receive *q* acceptances, it will generally have to offer to admit more than *q* applicants. The problem of determining how many and which ones to admit requires some rather involved guesswork. It may not be known (a) whether a given applicant has also applied elsewhere; if this is known it may not be known (b) how they rank the colleges to which they have applied; even if this is known, it will not be known (c) which of the other colleges will offer to admit them. A result of all this uncertainty is that colleges can expect only that the entering class will come reasonably close in numbers to the desired quota and be close to attainable optimum in quality."[1]

In their paper, the authors stated that one elaboration could be introducing a waiting list, where a candidate cannot be informed that he has not been admitted but may be admitted later if the vacancy occurs. However, they identified this could lead to some other new problems where the applicant could take a position of playing safe, accepting the first offer they receive even when it is not their favorite.

Here is when we use the term marriage because one set can be considered to be men and the other set women (thinking of straight relationships). In the stable marriage problem, each member of a set ranks the other set members in order of preference. Each member of a set ranks the other set members in order of preference. The assignment is regarded as *unstable* if the men are allocated to the women in such a manner that there is a man and a woman who are not assigned to each other but would both prefer each other over their current companions. On the other hand, the marriage assignment is stable if there is no such unstable assignment.

A. Motivation

The Stable Marriage Problem is not a regular computer science problem; it is a unique non-numeric mathematical and game theory problem that has been applied in multiple disciplines and areas, such as:

- *Education:* Matching students with schools or universities; used to match medicine student residents with hospitals.
- *Medicine:* Helping transplant patients to find a matching donor.
- *Computer Science:* Mostly used in large distributed internet services, where the problem is associated with the assignment of users to available servers.
- *Labor Market:* Matching employees with employers.

This paper will present and explain the problem alongside the solution algorithm, dive deeper into some real-life applications where the problem is being approached nowadays, and an extrapolation case where it can be used.

II. BASIC CONCEPTS

A. The Concept of Stable Matching

The stable matching problem involves a group of agents-disjoint sets-, let us call them A and B, and each agent ($a_i \in A$ & $b_i \in B$) has a preference list of individuals with whom he may be matched. A matching with the property that no pair of agents that are eligible to each other prefer another agent over their counterparts is a solution to such a challenge. *A stable matching* is the term for this type of match.

In a stable matching scenario, no agent can improve upon its assignment by contacting an agent they prefer to its already assigned counterpart. If this situation occurs, any such contacted agent will refuse the proposal since they choose the agent with whom they already are currently paired.

Additionally, the stable matching problems can be designed to allow or forbid ties in the preference lists, opening the door to multiple variations to the main problem [2]. When tie preferences are permitted, the concept of stability is defined as the absence of an unmatched pair of agents that strictly prefer each other to their assigned counterpart.

For simplicity, this paper does not explore in-depth problems where tie preferences are allowed; nonetheless, we will explore diverse variations where unties preferences are the only one allowed.

B. The Concept of Optimal Matching

In general, for each given set A and B, there exist multiple stable marriage solutions that will produce some optimality depending on which side the matching benefits [2]. With that being said, we can observe the following:

Set A optimal solution (left-side optimal solution). This is the stable solution when each member of the Set A ($a_i \in A$) is at least as well off under it as under any other stable solution, giving it stated list of preferences. It is mentioned to be left-side optimal solution since if we state our two sets in order to approach the problem, members of *set A* will be the ones taking the first decision.

Set B optimal solution (right-side optimal solution). This is the stable solution when each member of the Set B ($b_i \in B$) is at least as well off under it as under any other stable solution, giving it stated list of preferences. It is said to be right-side optimal solution since if we state our two sets in order to approach the problem, members of *set B* will be the ones taking the second decision, accepting or rejecting the left-side proposals.

Minimum choice Stable Solution: In this stable solution the sum of the choice numbers of the Set A and members of the Set *B* is a minimum. Although this method isn't unique, it does give a type of selfless optimum solution by crediting low choice numbers in both groups.

All the three solutions might be the same, or the minimum choice stable solution may coincide with either the set A optimal solution or the set B optimal solution [2].

We can exemplify the previously mentioned. Let us consider a simple matching/marriage assignment problem with the same cited sets, A and B. A is composed of a1,a2, and a3; and B is made of b1,b2,b3. Members of set A prefer members of set B in the following rank:

Set A members	Ranks 1st	Ranks 2nd	Ranks 3rd
A_1	B_1	B_2	B_3
A_2	B_2	B_1	B_3
A_3	B_1	B_3	B_2

And the members of set B rank members of set A in the following order:

Set B members	Ranks 1st	Ranks 2nd	Ranks 3th
B1	A_2	A_1	A_3
B2	A_3	A_2	A_1
B3	A_1	B_3	B_2

The matches of A_1&B_1, A_2&B_2, and A_3&B_3 produce a stable marriage because only one member of set A, A_3 would consider another assignment something better (B_1 over B_3), and from set B, B_1 prefers A_2 over A_1. This example represents the set A optimal solution since as we can see, most of the Set A members got paired with their first ranked option. On the other hand, we could generate the scenario where we get the Set B optimal solution, being A_1&B_3, A_2&B_1, and A_3&B_2, where all members of set B get their option ranked at the first place. Summing the amount choices members of set A have, we count 10, in comparison with the set B choices which is 11. Therefore, this also fulfills the condition that the minimum choice stable solution is the same as the set A optimal solution (left-side optimal solution) [2].

III. GALE-SHAPLEY ALGORITHM

A. Algorithm

D. Gale and L. S. Shapley developed a game theory algorithm where there always exists at least one stable matching in an instance of the stable marriage problem [1]. The authors studied two disjoint sets equally sized (n) and used the analogy of *men* and *women*. As mentioned, each person of each set creates a strictly ordered list stating their preferences of all the members of the opposite sex (thinking of straight relationships); therefore, person p prefers q to r, where q and r are part of the set of the opposite sex to p, if and only if q precedes r on p's preference list [1].

The Gale-Shapley algorithm always finds a stable marriage solution, which, as previously established, is uniquely beneficial to set A (let us call it men) or set B (let us call it women), depending on the roles of the two groups during the algorithm execution. With all of the assumptions established, this algorithm may be viewed as a series of male-to-female proposals. During the execution of the algorithm, each member of each set is either engaged or free (paired or not paired), but once a woman is engaged, she will never be free. A guy who engages several times, on the other hand, acquires couples that are less appealing to him, but each subsequent engagement gives a lady a more favored partner.

When a free woman accepts a proposal, she will take it right away and engage the proposer. When an engaged woman receives a proposal, she compares the proposer to her current partner and rejects the less favored of the two men; in other words, if she prefers her current partner, she rejects the new proposal; however, if she prefers the new proposer, she breaks her current engagement and engages to the current proposer, freeing her ex-partner [1].

On the other hand, each man proposes to the women on his ranked list, starting from his first choice to the last one, until he becomes engaged. If a man gets engaged and then that engagement is broken, he becomes free again and resumes the proposal procedure according to his list. The algorithm ends when all members of both parties are engaged.

We can represent the cited algorithm with the following pseudocode:

```
1. Each individual ranks the opposite sex
2. Assign each person to be free
3. while there is an unmarried man do
4.     man chooses the first woman on his preference list
       he has not proposed to yet and proposes to her
5.     If woman is unmarried or prefers man over her
       current partner (man0) then
6.         woman divorces man0
7.         woman marries man # (new proposer)
8. Output the stable matching consisting of n engaged pairs
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Fig. 1. Basic Gale-Shapley Algorithm

The fundamental concept of the original version of Gale-Shapley algorithm was stated on the following theorem:

Theorem: *For any given instance of the stable marriage problem, the algorithm terminates, and on termination, the engaged pairs, constitute a stable matching.* [5]

This theorem is easily proof, as at first we assumed that no man can be rejected by all the women. A woman can reject only when she is engaged, and once she is engaged she won't become free ever again. Thus, the rejection of a man by the last woman on his list would imply that all the women were already engaged. But since equally sized sets, and no man has more than one pair, all the men would also be paired, which is a contradiction. [5]

As no man can propose to more than one woman at each iteration, and no man can propose more than once to the same woman, then the total number of iterations can't surpass n^2, and therefore having a time complexity of $O(n^2)$.

B. Simple Example

Consider we have two sets of men and women of the same size. For graphical purposes, let us say our men's set is composed of A, B, and C, and our women's set is composed of α, β, and γ. Each man and woman create a list where they rank each of the people of the opposite sex planning to get married in the future. Each set's ranking list is as follows:

Men	Men's Ranking		
	Ranked 1st	Ranked 2nd	Ranked 3rd
A	β	γ	α
B	α	γ	β
C	α	β	γ

Fig. 2. Example Gale-Shapley Algorithm: Men's ranking

Women	Women's Ranking		
	Ranked 1st	Ranked 2nd	Ranked 3rd
α	A	B	C
β	B	C	A
γ	C	B	A

Fig. 3. Example Gale-Shapley Algorithm: Women's ranking

After having their ranking list and determining each member as free agents, then the algorithm starts iterating. For the first iteration, A proposes to β, B proposes to α, and C presents to α, leaving γ without receiving any proposals at the time. As all members of the women's set are unmarried/engaged, they decide. β only received a proposal coming from A, who is the third stable option she would accept; however, in the case of α, given she has received proposals from B and C, she has to decide with whom to stay. According to her ranking list, she (α) chooses to engage with B, setting C free to propose again in the next iteration.

Because C is unmarried, then we run the loop again, so C proposes to his second-best stable companion who is β. β, who is already engaged needs to decide whether to stay with A (who is her actual pair), or engage with C. To make this decision, β checks her list and decides to break up with A and engage with C, making A now available to propose again in the next iteration.

Now C checks his list and propose to his third-best option γ, who was unmarried. As everyone now is married/engaged the algorithm stops.

M	1st	2nd	3rd
A	~~β~~	**γ**	α
B	**α**	γ	β
C	~~α~~	**β**	γ

W	1st	2nd	3rd
α	A	**B**	C
β	B	**C**	~~A~~
γ	C	B	**A**

Fig. 4. Example Gale-Shapley Algorithm: Final results after running the algorithm

The resulted matches after running the algorithm are (A, γ), (B, α), (C, β). This algorithm is also called the deferredacceptance" algorithm, because of the engagement and possible future process of unengaged towards pairing to a better suitor.

IV. Other Marriage/Matching Problems

We already know about the stable marriage problem, that works using to sets of the same size; there are also some interesting variants of the problem, such as the roommate problem, the intern assignment problem, and the intern assignment problem with couples.

A. The Roommate Problem

Instead of two equally sized sets, the Roommate Problem focuses on a single group of members of even cardinality *n* (even number of elements). Each member has a preference list over the other (making a ranking list of n-1 agents). To be considered a stable matching for this problem, we look for a split of the single set we have into *n*/2 pairs so that paired members both prefer each other over their partners under the matching. There exist cases of the roommate problem for which no stable matching exists, both with and without ties, as we previously defined the concept of stable matching. These rules are inherited from the conditions Gale and Shapley noted. [3] Illustration of the roommate problem:

Person	Ranked Preference list
A	BCD
B	CAD
C	ABD
D	Arbitrary

Fig. 5. Illustration of the Rommate Problem

Anyone assigned to D will find a person he prefers and who prefers him [3].

Robert W. Irving, academic from University of Glasgow, established an efficient algorithm for determining if a stable assignment exists for each instance of the roommate problem without ties [6]. When ties are allowed, the roommate problem is NP-complete.

B. The Intern Assignment Problem

The intern assignment problem is, in a glance, a "polygamous" version of the stable marriage problem, in which individuals of one gender can accept up to a predetermined number of partners of opposite gender [3]. This is one of the most interesting variants since it used every year by the National Residency Matching Program (NRMP) to assign graduating medical students looking for their medical residency to hospital which are looking to hire new medical residents [8].

Prior to the usage of this algorithm, hospital benefited from filling positions as early as possible, and applicants used to take their time to accept an offer (in this case the graduating medical students), being a contrary relationship between both interests [8]. This problem presents that each hospital has a defined number of positions (quota) it wants to fill and a priority list that ranks the interns according to their preferences over who they want to hire to fill those vacancies. A hospital can rank in its preference list the possibility of leaving a position unassigned over being assigned to some of the graduating medical students.

On the other hand, each graduating medical student has a ranking list of possible hospitals that defines which hospitals the graduate would want to be allocated to. A potential medical resident might rank the chance of being unassigned to anybody from a list of undesirable hospital positions in his rating list. Therefore, future medical residents can leave hospitals out of their preference list [3].

If one of the following conditions occurs, then this matching process is unstable:

- There exists a hospital or an intern may decide to remain unassigned rather than accept a matched assignment.
- There exists a hospital and a graduate such that the graduate prefers this hospital over the one to which the medical student had been assigned, and at the same time, this hospital likes this graduate over the current assignment.

The biggest differences compared to the stable marriage problem Gale and Shapley approached are that the sets don't have to be of the same size, one member of the parties can decide to not choice any specific agent from the counterpart set, and the hospitals (in this case right-side) can decide to choose more than one agent, leading to the "polygamous" factor.

To better understand this problem, let us consider two sets, one of the graduate medical students looking for medical residency and the other of the hospital hiring them. The list of students is composed of A, B, C, D, and E; and the list of participating hospitals is composed of α, β, and γ. Each of the listed hospitals has a total of two (2) vacant positions, so they only can pick up to two medical residents.

Graduate Medical Students' preference List of Hospitals

Rank	A	B	C	D	E
1	β	β	β	α	β
2		α	γ	β	α
3				γ	γ

Fig. 6. Illustration of the Intern Assignment Problem: List of Graduate Students' Preference list ranked

Hospitals' Preference List of Graduate Medical Students

Rank	α (2 positions)	β (2 positions)	γ (2 positions)
1	E	E	E
2	C	A	A
3		B	C
4		D	D
5		C	

Fig. 7. Illustration of the Intern Assignment Problem: Hospitals' Preference List of Graduate Medical Students

This process behaves similar to the Gale-Shapley algorithm in the sense that students propose to hospitals, and further, each hospital decides to keep the current medical resident or pick a better one according to their ranked list. If a student is free, then the algorithm decides to assign them to the next hospital on their list. If there are no more hospitals on their list, then for this graduate medical resident, the algorithm stops; however, the algorithm will stop if no more stable matches are available.

For our illustration, the first iteration tentatively assigns A to β and B to β, being β their primary option. Then when it tries to assign C to β realizes that A and B are already assigned to β, and they rank better than C, so the algorithm attempts to rank C to γ, being the second option. Later, the algorithm tries to match D with α; however, as α did not consider D on its list, then it moves to attempt matching D with β, but β is already assigned to A and B (which both rank better), so it assigns D to the third option which is γ, that only has one unfilled position. Finally, when E's turn comes, and the algorithm checks its list of preferences, the algorithm assigns E to its first option, β, which has ranked E as its main option. Being E assigned to β, β frees its worst possible candidate because only two (2) spots are available. In this case, the freed candidate is C.

The algorithm confirms E with β since both rank 1st on each other's list. There is no chance E will be displaced in further iterations. As B is now α, the algorithm attempts to match it to its next option, α; nonetheless, α did not rank B, leaving B without a match, and also α stays without any medical resident since there are no more stable matches available. The algorithm stops.

Matching Results

No Match

Resident	B
Hospital Rank	**α**
1	E
2	C

Matches

Rank / Hospital	**β**	**γ**
1	**E**	E
2	**A**	A
3	~~B~~	**C**
4	~~D~~	**D**
5	~~C~~	

Fig. 8. Illustration of the Intern Assignment Problem: Final results

Each strikethrough entry represents a medical student resident reassignment.

C. *The Intern Assignment Problem with Couples*

The National Residency Matching Program allows graduate medical students to apply as a couple instead of as individuals [10]. The complexity of the problem changes as now it is considering joint preference lists over pairs of positions. This coupling event corresponds to the possibility that some of the medical graduates are married to each other and desire to be assigned to hospitals that are geographically close to each other. However, as with the Intern Assignment Problem, the existence of a solution for every occurrence of the problem is not guaranteed [3].

For this problem, there are instances with tie and without a tie; however, there are no efficient methods to determine whether a stable solution exists, neither with nor without a tie. These sorts of scenarios can be found more detailed in Roth's work [9], and a demonstration of its NP-complete complexity is found in Ronn [3].

V. A Problem that has Change the World of Medicine

Not only the American scene of Medicine and Education has benefited from the usage of algorithms that solves game theory problems, but in 2004, Alvin Roth [11] developed the principle to help transplant patients to find donors. Previously, less than 20 patients per year got kidneys from living donors, despite the fact that transplants from living donors result in significantly improved patient outcomes. The issue was simple: many people wanted to give a kidney to a loved one but couldn't because of blood type and other characteristics that made them incompatible. Then Alvin Roth developed an exchange system based on the Gale-Shapley algorithm to assist incompatible donor-recipient pairings in finding others in the same situation. The implementation of this system is helping thousands of people on the cited conditions and made him earned an Nobel Prized in 2012 alongside Lloyd Shapley [12].

VI. Extrapolation

A. *Description of an actual problem*

The Stable Marriage Problem can be extended to be applied in multiple areas where there is still missing an effective matching procedure o method. Nowadays, the Fine Arts High Education system of the Dominican Republic has a similar problem that the education and medical residents system in the United States used to have before the 1960s. The Fine Arts High Education schools benefit from filling new professor vacancies as early as possible. These professors must be individuals who graduated from the Fine Arts Education system, whether from public elementary fine arts schools or the system itself. On the other hand, students who want to apply to these schools benefit from delaying acceptances as there are multiple academies and the desire to make the best decision feasible. Given these factors, several problems arise, like offers being made for teaching positions 2 years before graduating.

Because of this, some of the art schools decided only to submit to the Ministry of Culture letters of recommendation during the students' last year, attempting to give the opportunity to the students who are close to graduation to work in this area. Nonetheless, because of competition, some of these schools started implementing time-limited offers leading these future professors to decide within a short period of time (accept or reject). Very high-skilled and competitive applicants tend to hold offers as much as possible, rescinding the current ones when a better one comes along.

These factors have led to the problem where multiple schools in the Dominican Fine Arts Education System do not admit more students because of the lacking of professors; however, there are numerous graduates from the system that can't find a job after graduation. Furthermore, some of these schools have a waiting list that can take up to 18 months to admit new students because there is an overpopulation of students compared to the hired professors [13].

B. Solution

A plausible solution to this problem could be implementing a similar system to the one used by the National Residency Matching Program. The Ministry of Culture of the Dominican Republic can designate a specialized office responsible for applying this version of the algorithm to match graduating fine arts students to schools so fine arts schools in the Dominican Republic can fill out vacancies more evenly and therefore could have the capacity admit more students.

VII. Conclusions

As seen, the stable marriage/matching problem is subtly applied in multiple areas; even though it is not a computational problem, being a game theory problem; however, it can be computationally implemented. The search for its solution has produced enormous benefits not only in our daily tasks, such as the case of its usage by servers, or helping students and graduates medicine residents to match schools and hospitals, but also saving thousands of lives by connecting compatible organ donors to patients. Implementing solutions to this problem in other disciplines is vital, as we learned from its diverse actual applications, and there are still tremendous opportunities in multiple fields where they can be applied.

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